AN ESTIMATE FOR THE NUMBER OF ZEROES OF ANALYTIC FUNCTIONS IN n-DIMENSIONAL CONES

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1. INTRODUCTION

The relation between the order of growth of an entire function in $\mathbb{C}^n$ and the area of its zero-variety, and more generally Nevanlinna theory in several complex variables, has been extensively studied in the recent past by Chern, Griffiths, Lelong, Stoll, among others (see, e.g., [12] for references). The techniques used by these authors are essentially similar to the differential-geometric method employed by Nevanlinna and Ahlfors in the case of a single variable.

Many problems in analysis require a similar extension (from one to several variables) of results known for functions defined in angular regions of $\mathbb{C}^1$. For reasons that will become apparent below, it is not possible to reduce the problem to the one-variable case; nevertheless, using a potential-theory approach one can still obtain the required estimates (Theorem 2 of §4 below).

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2. PRELIMINARIES

Let us recall some standard notation (cf. [7]). The exterior derivative in $\mathbb{C}^n$ can be written as $d = \partial + \bar{\partial}$, and with $d^c = \frac{i}{4\pi}(\bar{\partial} - \partial)$ we obtain

$$dd^c = \frac{i}{2\pi} \partial\bar{\partial}.$$

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In $\mathbb{R}^m$ we indicate by $\Delta = \Delta_m$ the Laplace operator, $\Delta g = \sum_{j=1}^{m} \frac{\partial^2 g}{\partial x_j^2}$, so it makes sense to apply Δ_{2n} to functions of n-complex variables by identifying $\mathbb{C}^n = \mathbb{R}^{2n}$.

If $z = (z_1,\cdots,z_n) \in \mathbb{C}^n$, $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$, we write $B(0,r) = B_r = \{\|z\| \leqq r\}$, $S_r = \{z : \|z\| = r\}$ for $0 < r < \infty$. More generally, $B(a,r) = \{z : \|z-a\| \leqq r\}$. We can define two (1,1)-forms ϕ, ψ by

$$\phi = dd^c\|z\|^2 = \frac{i}{2\pi} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$$

$$\psi = dd^c \log\|z\|^2, \qquad z \neq 0.$$

Then $\phi_n = \phi \wedge \cdots \wedge \phi$ (n times) is the volume form of $\mathbb{C}^n$, and more generally the restriction of ϕ_k to any k-dimensional (complex) linear variety is the euclidean area form of the variety. On the other hand, ψ_{n-1} is a measure of "projective" area: it is invariant under unitary transformations and complex dilations, and

(1) $$\omega_{2n-1} = d^c \log\|z\|^2 \wedge \psi_{n-1},$$

is the area form in the unit sphere $S_1 = \{\|z\| = 1\}$, $\int_{S_1} \omega_{2n-1} = 1$. If f is an analytic function, then $\log|f(z)|$ is subharmonic, i.e. $\Delta_{2n} \log|f(z)|$ defines a positive measure, whose support is the analytic variety $V = \{z : f(z) = 0\}$. Moreover,

(2) $$dd^c \log|f(z)|^2 \wedge \phi_{n-1} = \tfrac{1}{2}(\Delta \log|f(z)|)\phi_n;$$

it follows that the l.h.s. of (2) defines the euclidean area form in V. As usual, we can define the <u>counting</u> <u>function</u> by

$$\sigma(r) = \int_{B_r} dd^c \log|f(z)|^2 \wedge \phi_{n-1} \qquad 0 < r < \infty.$$

More usually, if D is a cone in $\mathbb{C}^n$ (having vertex at the origin) and $D_r = D \cap B_r$, then

(3) $$\sigma_D(r) = \int_{D_r} dd^c \log |f(z)|^2 \wedge \phi_{n-1}.$$

Similarly, we have the projective area of V, defined

$$\nu(r) = \int_{B_r} dd^c \log |f(z)|^2 \wedge \psi_{n-1}.$$

If we assume further that $f(0) \neq 0$, we have the following crucial formula in Nevanlinna theory

(4) $$\nu(r) = \frac{\sigma(r)}{r^{2n-2}}$$

<u>Sketch of the proof</u>. Clearly $d\phi_{n-1} = d\psi_{n-1} = 0$. Furthermore, one sees easily that $\psi_{n-1} = \|z\|^{-2n+2}\phi_{n-1}$; hence, by Stokes theorem,

$$\nu(r) = \int_{B_r} dd^c \log|f(z)|^2 \wedge \psi_{n-1} = \int_{S_r} d^c \log|f(z)|^2 \wedge \psi_{n-1}$$

$$= \int_{S_r} d^c \log|f(z)|^2 \wedge \frac{\phi_{n-1}}{r^{2n-2}} = \frac{1}{r^{2n-1}} \sigma(r).$$

<u>Remark 1</u>. This simple relation fails when σ is replaced by σ_D due to the appearance of additional boundary terms.

<u>Remark 2</u>. For $n = 1$, $\sigma(r) = \nu(r) =$ number of zeroes of f in B_r.

The next important formula allows us to compute $\nu(r)$ by reducing it to the one variable case. It is <u>Crofton's formula</u> [11]

(5) $$\nu(r) = \int_{\xi \in S_1} \omega_{2n-1}(\xi) \int_{|\lambda| \leq r} dd^c \log|f(\lambda\xi)|^2,$$

where the operator dd^c acts on the complex variable λ, so the inner integral just counts the number of zeros of $g(\lambda) = f(\lambda z)$ in $\{|\lambda| \leq r\}$.

Let us recall that a function f is said to be of <u>order</u> ρ $(\rho > 0)$ and <u>finite type</u> if there exist constants $A, B > 0$ such that

$$|f(z)| \leq A \exp\{B\|z\|^{\rho}\}.$$

For such functions, it is known (cf. [9, p.44]) that

$$\overline{\lim_{r\to\infty}} \frac{1}{r^{\rho}} \int_{|\lambda|\leq r} dd^c \log|f(\lambda z)|^2 \leq e^{\rho}B$$

and therefore by (4) and (5)

(6) $$\overline{\lim_{r\to\infty}} \frac{\sigma(r)}{r^{\rho+2n-2}} = \overline{\lim_{r\to\infty}} \frac{\nu(r)}{r^{\rho}} \leq e^{\rho}B.$$

Similarly, Crofton's formula shows that if f is a polynomial of degree m, then $\nu(r) \leq m$.

We now recall two theorems from the theory of functions of one complex variable. Let g be an analytic function defined in the half-plane $\{\mathrm{Re}\,\lambda \geq 0\}$, of order ρ and finite type, such that $g(0) \neq 0$. Denote by $\nu_g(r)$ the number of zeroes of g in the disk $\{\lambda : |\lambda - r/2| \leq r/2\}$.

THEOREM I. [9, p.185] *If* $\rho > 1$ *then there exists an increasing function* $s_g(\theta)$ *such that*

$$\lim_{r\to\infty} \frac{\nu_g(r)}{r^{\rho}} \leq \frac{(1+1/\rho)^{\rho}}{2\pi(\rho-1)} \int_{-\pi/2}^{\pi/2} \cos^{\rho}\theta \, ds_g(\theta) \leq C_{\rho}B$$

where C_{ρ} *is a positive constant independent of* g *and* B *is the constant involved in the definition of finite type.*

Remark 3. By using conformal mappings, we can obtain a similar theorem for functions of order $\rho > \pi/\alpha$, defined in the angle $|\arg\lambda| \leq \alpha/2$. This possibility does not exist in $\mathbb{C}^n$, $n \geq 2$, since by a theorem of Liouville the only conformal maps are the Möbius transformations.

The generalization of theorem I to cones in $\mathbb{C}^n$ is the objective of this paper and appears in §4.

Suppose f is holomorphic of order ρ and finite type, in an open cone D in $\mathbb{C}^n$. We define the <u>indicator function</u> of f by

(7) $$h^*(z) = \overline{\lim_{\substack{y\to z\\ y\in D}}}\ \overline{\lim_{r\to\infty}} \frac{\log|f(ry)|}{r^\rho} \qquad z \neq 0.$$

This function is (pluri)-subharmonic and homogeneous of degree ρ. For $n = 1$, the outer $\overline{\lim}$ is not necessary and the function h^* is even continuous.

We say f is of <u>completely regular growth in</u> D if for almost all $z \in D \cap S_1$, we have

(8) $$h^*(z) = \lim_{r\to\infty} \frac{\log|f(rz)|}{r^\rho} .$$

Then we have the following

<u>THEOREM II</u>. [9, p.182] *Let* g *be an analytic function of order* $\rho > 0$ *and completely regular growth in* $\{\lambda \in \mathbb{C}^1 : \mathrm{Re}\,\lambda \geq 0\}$. *Then there exists an increasing function* $s_g(\theta)$ *such that*

(9) $$\lim_{r\to\infty} \frac{\nu_g(r)}{r^\rho} = \frac{1}{2\pi\rho} \int_{-\pi/2}^{\pi/2} \cos^\rho\theta \; ds_g(\theta) < \infty.$$

The meaning of ν_g is the same as in Theorem I. The generalization of formula (9) to several variables is due to Gruman [7].

3. FUNCTIONS OF COMPLETELY REGULAR GROWTH

We assume the number of variables is $n \geq 2$. If $N \subset S_1$, we define N_∞ to be the cone generated by N, i.e.

(10) $$N_\infty = \{tz : z \in N, \ t > 0\},$$

and as before $N_r = N_\infty \cap B_r$.

Using the method of L. Gruman, we prove the following result.

PROPOSITION 1. *Let* p *be a non-zero polynomial,* N *an open set* $\subset S_1$, *and* f *a function analytic in* N_∞ *such that for every compact* $K \subset N$, *we have*

$$f(z) = p(z) + O(\|z\|^{-1}) \tag{11}$$

uniformly in K_∞. *Then*

$$\overline{\lim_{r\to\infty}}\ \sigma_{K_\infty}(r)\ \frac{r^{2-2n}}{\log r} < \infty. \tag{12}$$

Proof. If $z \in S_1$, $t > 0$, $p(tz) = t^m p_m(z) + O(t^{m-1})$, where p_m is a homogeneous polynomial of degree m. Clearly both p and f are of completely regular growth in the sense that if $z \in N$ and $p_m(z) \neq 0$ then

$$\lim_{r\to\infty} \frac{\log|f(rz)|}{\log r} = \lim_{r\to\infty} \frac{\log|p(rz)|}{\log r} = m. \tag{13}$$

Take any such $z \in N$ and pick ε, $0 < \varepsilon < 1$, such that $D' = \{w \in S_1 : \|w-z\| < \varepsilon\} \subset N$. Let $D = \{w \in S_1 : \|w-z\| < \varepsilon/2\}$. For almost all $s > 0$ we have $f(sz) \neq 0$, so from Crofton's formula one obtains the Jensen formula in n-variables

$$\int_{S_1} \log|f(s(z+\varepsilon\zeta))|\,\omega_{2n-1}(\zeta) - \log|f(sz)| = \int_0^{\varepsilon s} \sigma_{sz}(t)\,\frac{dt}{t^{2n-1}}, \tag{14}$$

where $\sigma_{sz}(t) = \int_{B(sz,t)} dd^c \log|f(w)|^2 \wedge \phi_{n-1}$.

The right hand side of (14) satisfies

$$\int_0^{\varepsilon s} \sigma_{sz}(t)\,\frac{dt}{t^{2n-1}} \geq k_1(\varepsilon)\,\frac{\sigma_{sz}(3/4\ \varepsilon s)}{s^{2n-2}},$$

where $k_1(\varepsilon)$ is a positive constant.

In other words, for any $r > 1$ we have

$$(15)\qquad k_1(\varepsilon)\int_1^r \sigma_{sz}(\tfrac{3}{4}\varepsilon s)\frac{ds}{s^{2n-1}} \leqq \int_{S_1}\omega_{2n-1}(\zeta)\int_1^r \log\frac{|f(zs+\varepsilon s\zeta)|}{|f(sz)|}\frac{ds}{s}\,.$$

Since $r > 1$, we can find an integer $m \geq 1$ such that

$$(16)\qquad (1+\varepsilon/4)^m \leqq r < (1+\varepsilon/4)^{m+1}.$$

Define

$$(17)\qquad a_q = (1+\varepsilon/4)^q \qquad q = 0,\cdots,m.$$

From the definition of D it follows that for $a_{q-1} \leqq s < a_q$

$$(18)\qquad D_{a_q}\setminus D_{a_{q-1}} \subset B(sz, \tfrac{3\varepsilon}{4}s) \qquad q = 1,\cdots,m.$$

Hence

$$(a_q - a_{q-1})\int_{D_{a_q}\setminus D_{a_{q-1}}} dd^c\log|f(w)|^2\wedge\phi_{n-1}$$
$$= (a_q - a_{q-1})\int_{a_{q-1}}^{a_q} d\sigma_{D_\infty}(s)$$
$$\leqq a_q^{2n-1}\int_{a_{q-1}}^{a_q}\sigma_{sz}(\tfrac{3\varepsilon}{4}s)\frac{ds}{s^{2n-1}}\,.$$

Therefore

$$a_{q-1}^{2n-1}\left(\frac{a_q}{a_{q-1}} - 1\right)\int_{a_{q-1}}^{a_q} s^{2-2n}d\sigma_{D_\infty}(s) \leqq a_q^{2n-1}\int_{a_{q-1}}^{a_q} s^{1-2n}\sigma_{sz}(\tfrac{3}{4}\varepsilon s)\,ds,$$

and we obtain

$$(19)\qquad \frac{\varepsilon}{4+\varepsilon}\int_{a_{q-1}}^{a_q} s^{2-2n}\, d\sigma_{D_\infty}(s) \leq \int_{a_{q-1}}^{a_q} s^{1-2n}\, \sigma_{sz}(\tfrac{3}{4}\varepsilon s)\, ds.$$

By adding the inequalities in (19) for $q = 1,\cdots,m$ and using (15) and (16) we obtain with a new constant $k_2(\varepsilon) > 0$

$$(20)\qquad k_2(\varepsilon)\int_{a_0}^{a_m} \frac{d\sigma_D(s)}{s^{2n-2}} \leqq \int_{S_1} \omega_{2n-1}(\zeta) \int_1^r \log \frac{|f(sz+s\varepsilon\zeta)|}{|f(sz)|}\,\frac{ds}{s}\ .$$

From (11) it follows that for $w \in D'_\infty \setminus D'_1$

$$\log|f(w)| \leqq m \log\|w\| + O(1)$$

$$\log|f(sz)| = m \log s + O(1).$$

Therefore the integral on the right hand side of (20) can be bounded by (constant) log r; the left hand side can be integrated by parts, and we finally obtain

$$\frac{\sigma_D(r)}{r^{2n-2}} \leqq k_3 \log r + k_4,$$

where k_3, k_4 are positive constants depending on ε and z. We obtain a similar inequality for any compact $K \subset N$, by choosing a finite covering of K by sets D as above.

Remark 4. From (11) it follows that the analytic variety V defined by f in $K_\infty \setminus K_R$ lies within an ε-neighborhood of the variety $V_p = \{z : p(z) = 0\}$ for R sufficiently large. It follows from a theorem of Rudin [10] that if an analytic variety V in $\mathbb{C}^n$ lies within an ε-neighborhood of an algebraic variety then it is itself algebraic and therefore $\nu(r)$ is bounded. Additional assumptions on the function f of Proposition 1 should enable one to eliminate the factor $\log r$ from the conclusion; for example, one might assume that for any $z,\zeta \in \mathbb{C}^n$, $\|\zeta\| = 1$, the number of zeroes of $g(\lambda) = f(z+\lambda\zeta)$ $(\lambda \in \mathbb{C})$ in a disk of radius 1 is bounded independently of z and ζ.

An example of a function f with the property mentioned in the above remark is the exponential polynomial,

$$f(z) = \sum_{j=1}^{\ell} a_j(z) \exp\langle z,\alpha_j\rangle \qquad \ell \geqq 2. \tag{21}$$

Here the a_j are non-zero polynomials, $\alpha_j = (\alpha_{j,1},\cdots,\alpha_{j,n}) \in \mathbb{C}^n$ are distinct and $\langle z,\alpha_j\rangle = \sum_{k=1}^{n} z_k\alpha_{j,k}$. (cf. [13].) In this case, f is of completely regular type of order 1 with indicator function

$$h^*(z) = \max_j \operatorname{Re}\langle z,\alpha_j\rangle. \tag{22}$$

Then by Crofton's formula

$$\lim_{r\to\infty} \frac{\sigma(r)}{r^{2n-1}} = \int_{S_1} h^*(z)\omega_{2n-1}(z). \tag{23}$$

As a corollary of Proposition 1 we obtain the existence of regions with very few zeroes for f; it is enough to take

$$N^{(k)} = \{z \in S_1 : h^*(z) = \operatorname{Re}\langle z,\alpha_k\rangle > \max_{j\neq k} \operatorname{Re}\langle z,\alpha_j\rangle\}$$

and $f_k(z) = e^{-\langle z,\alpha_k\rangle} f(z)$ in $N_\infty^{(k)}$ (cf. [2]).

4. GENERAL CASE

Taking into account formulas (2) and (3), we can reduce the problem of estimating the number of zeroes of a non-zero analytic function in a cone D to estimating $\sigma_D(r) = \int_{D_r} \Delta u$, the so-called Riesz mass of the subharmonic function u. Though the method used below is more general†, we shall restrict ourselves to circular cones in $\mathbb{R}^m$.

By $x = (x_1,\cdots,x_m)$, $|x|$ we denote respectively a point in $\mathbb{R}^m$

† e.g. if $D \cap S_1$ has a smooth boundary with bounded curvature.

and its euclidean norm. To keep the notation uniform we will assume $m \geqq 3$, though the case $m = 2$ is easier to deal with. We introduce polar coordinates $(r,\theta_1,\cdots,\theta_{m-1})$ by

$$(24)\qquad 0 < r = |x|, \quad x^* = x/r, \quad \theta_1 = \text{arc cos } x_1^* \qquad (0 \leqq \theta_1 \leqq \pi)$$

where the remaining θ's are defined in the usual manner. Then the Laplacian can be written as

$$(25)\qquad \Delta_m = \Delta = r^{1-m}\frac{\partial}{\partial r}(r^{m-1}\frac{\partial}{\partial r}) + r^{-2}\delta,$$

where δ is an operator involving only the angular variables, namely the Laplace-Beltrami operator on the sphere $S_1 = \{x \in \mathbb{R}^m : |x| = 1\}$. There is only one case where we need an explicit description of δ. Assume the harmonic function v depends only on the coordinates r, θ_1, and that $v(r) = v(r,\theta_1) = r^\rho f(\theta_1)$, $\rho > 0$. Then

$$\Delta v = v_{rr} + \frac{m-1}{r} v_r + r^{-2} v_{\theta_1\theta_1} + (m-2) r^{-2}(\cot \theta_1)v_{\theta_1} = 0$$

or

$$(26)\qquad f''(\theta_1) + (m-2)\cot \theta_1 \, f'(\theta_1) + \rho(\rho+m-2)f(\theta_1) = 0.$$

By the change of variable $\xi = \cos \theta_1$, $f(\theta_1) = g(\xi)$, we have

$$(27)\qquad (1-\xi^2)g''(\xi) - (m-1)\xi g'(\xi) + \rho(\rho+m-2)g(\xi) = 0.$$

The solutions of (27) that are regular for $\xi = 1$ are the Gegenbauer functions, given explicitly [1, vol.3, p.276] by

$$(28)\qquad g(\xi) = C_\rho^{\frac{m-2}{2}}(\xi) = \frac{\Gamma(\rho+m-2)}{\Gamma(\rho+1)\Gamma(m-2)} F(\rho+m-2;-\rho;\tfrac{m-1}{2};\tfrac{1}{2}-\tfrac{1}{2}\xi),$$

where $F(\alpha;\beta;\gamma;t)$ denotes the hypergeometric function regular for $t = 0$. Furthermore, $g(1) = \frac{\Gamma(\rho+m-2)}{\Gamma(\rho+1)\Gamma(m-2)} \neq 0$ and it follows from a theorem of F. Klein (see [8, p.286]) that the function g has exactly $[\rho+1]$ zeroes in $(-1,1]$ if ρ not integral, and ρ zeroes if ρ is a positive integer (where $[s]$ = integral part of s).

A circular open cone $K(\alpha)$, $0 < \alpha < \pi$, is defined by the condition

(29) $$K(\alpha) = \{x \neq 0 : 0 \leq \theta_1 < \alpha\}.$$

Let $S(\alpha) = K(\alpha) \cap S_1 = \{x : r = 1, \ 0 \leq \theta_1 < \alpha\}$. Then the eigenfunctions f and eigenvalues μ of δ in $S(\alpha)$ are defined by the condition

(30) $$\delta f + \mu f = 0 \quad \text{in} \quad S(\alpha), \quad f = 0 \quad \text{on} \quad \partial S(\alpha).$$

Since δ is an elliptic operator, we obtain a sequence of eigenvalues $0 < \mu_1 < \mu_2 < \cdots$. Then we can write

(31) $$\mu = \rho(\rho+m-2), \quad \rho > 0.$$

Corresponding eigenfunctions can be found which are functions only of of θ_1, namely $f_\rho(\theta_1) = C_\rho^{\frac{m-2}{2}}(\cos\theta_1)$, where the ρ's are characterized by the condition

(32) $$C_\rho^{\frac{m-2}{2}}(\cos\alpha) = 0.$$

For instance, for $\alpha = \pi/2$ we obtain $\rho_1 = 1$, independent of the dimension m.

It follows from the above that for any $\rho > 0$, $\rho \neq \rho_1, \rho_2, \cdots$ we can find a harmonic function v_ρ in $K(\alpha)$ and a positive constant K_ρ with the properties

(33) $$|v_\rho(x)| \leq K_\rho r^\rho, \qquad v_\rho(x) = -r^\rho \qquad \text{for} \quad x \in \partial K(\alpha).$$

In fact, we can take v_ρ to be a constant multiple of f_ρ. For $\rho = \rho_n$, and α' sufficiently close to α $(0 < \alpha' < \alpha)$ we can similarly find harmonic functions v_ρ (actually depending also on α') in

$K(\alpha')$ such that the conditions above are satisfied with $\partial K(\alpha)$ replaced by $\partial K(\alpha')$.

Let us denote by $G(x) = g(x,a)$ the Green's function of $K(\alpha)$ with pole at a. Let ψ_n be the above eigenfunction with eigenvalue μ_n, normalized by the condition $\int_{S(\alpha)} |\psi_n|^2 \omega_{m-1} = 1$. Then following Bouligand [3], Lelong-Ferrand has proved that for $r > |a| = t$

$$G(x,a) = c_\alpha \sum_{n=1}^{\infty} t^{\rho_n} r^{\sigma_n} \frac{\psi_n(a^*)\psi_n(x^*)}{\sqrt{(m-2)^2 + 4\mu_n}}, \tag{34}$$

where $\sigma_n = -\rho_n - m + 2$, and c_α is the area of $S(\alpha)$ (see for instance [4] or [6]). From known estimates of these μ's and ψ's we can conclude that if $a \simeq (1,0,\cdots,0)$, $r \geqq 2$, then there exist positive constants k_1, k_2 such that

$$k_1 \leqq G(x)\, \mathrm{dist}(x^*, \partial S(\alpha))^{-1}\, r^{\rho_1+m-2} \leqq k_2 \tag{35}$$

and we have also

$$\frac{\partial G}{\partial r}(x) = \sigma_1 r^{\sigma_1 - 1} c_\alpha \frac{t^{\rho_1}\psi_1(x^*)\psi_1(a^*)}{\sqrt{(m-2)+4\mu_1}} + o(r^{\sigma_1-1}), \tag{36}$$

where $r \to \infty$. For $R \geqq 2$, the Green's function $G_R(x) = G_R(x,a)$ with pole at a of the region

$$K_R(\alpha) = \{x \in K(\alpha) : r < R\} \tag{37}$$

can be found to be

$$G_R(x) = G(x) - \left(\frac{R}{r}\right)^{m-2} G\left(\frac{R^2}{r^2}x\right). \tag{38}$$

Hence it follows from (36) that there exists a constant $k_3 > 0$

$$0 \leqq -\frac{\partial G}{\partial r} R(x)\Big|_{r=R} \leqq k_3 R^{\sigma_1 - 1}. \tag{39}$$

Finally, we need a lower bound on G_R on sufficiently large subsets of $K_R(\alpha)$. Take x such that $2 \leq r = \varepsilon R$, $0 < \varepsilon < 1$; we obtain from (35)

$$G_R(x) \geq \operatorname{dist}(x^*, \partial S(\alpha)) R^{-\rho_1-m+2}\left(\frac{k_1}{\varepsilon^{\rho_1+m-2}} - k_2 \varepsilon^{\rho_1}\right).$$

The expression in parentheses increases to $+\infty$ when $\varepsilon \to 0^+$. Therefore, there exists $\varepsilon_0 > 0$ such that for any r, R satisfying $2 \leq r \leq \varepsilon_0 R$,

$$G_R(x) \geq \frac{\operatorname{dist}(x^*, \partial S(\alpha))}{R^{\rho_1+m-2}} . \tag{40}$$

We can now prove our principal result.

THEOREM 1. *Let* u *be a subharmonic function in* $K(\alpha)$, *which is harmonic near* 0 *and satisfies* $u(x) \leq Br^{\rho}+C$ *for some positive constants* B, C, ρ. *Then for any* $0 < \beta < \alpha$, *we can find a constant* M, $M = M(u,\beta)$ *such that for* $R \geq 2$

$$\int_{K_R(\beta)} \Delta u \leq MR^{\rho^*+m-2}, \qquad \rho^* = \max(\rho, \rho_1).^{\dagger}$$

<u>Proof</u>. Let us leave aside for the moment the exceptional cases $\rho = \rho_1, \rho_2, \cdots$. Since $u \not\equiv -\infty$, the set $\{u = -\infty\}$ has measure zero and therefore we can find a point $a \in K(\alpha)$ as close as we want to $(1,0,0,\cdots,0)$ such that $u(a) \neq -\infty$. Applying Green's formula to the function $w = u - Bv_\rho - C$, v_ρ as defined in (33), we have for $R \geq 2$

$$\int_{K_R(\alpha)} G_R(x,a)\Delta u + w(a) = \int_{\partial K_R(\alpha)} w \frac{\partial G_R}{\partial n} \omega_{m-1} \tag{41}$$

where n denotes the inner normal. Clearly, $w \leq 0$ on $\partial K(\alpha)$;

†**for $\rho = \rho_1$ we have to take $\rho^* = \rho_1 + \varepsilon$, $\varepsilon > 0$.**

therefore, setting $S_R(\alpha) = \{x : x^* \in S(\alpha),\ |x| = R\}$, we have

$$\int_{\partial K_R(\alpha)} w \frac{\partial G_R}{\partial n} \leqq A(1+K_\rho)R^\rho \int_{S_R(\alpha)} \frac{\partial G_R}{\partial n} \leqq M_1 R^{\rho+\sigma_1-1} \int_{S_R(\alpha)} \omega_{m-1}$$

$$\leqq M_2 R^{\rho-\rho_1}.$$

These inequalities follow from (33), (39), and the definition of σ_1. Using (40), we conclude that

$$\int_{K_{\varepsilon_0 R}(\beta)} \Delta u \leqq M_3 R^{\rho+m-2} + M_4 R^{\rho_1+m-2} |w(a)| + \int_{K_2(\alpha)} \Delta u. \tag{42}$$

From here the conclusion of the theorem clearly follows, since the fact that u is harmonic near zero implies $\int_{K_2(\alpha)} \Delta u < \infty$. The exceptional cases are treated similarly by means of the construction in (33) above.

Remark 5. Clearly, the interesting case of theorem 1 occurs when $\rho > \rho_1$. If $\rho \leqq \rho_1$, the Phragmén-Lindelöf theorem (cf. [5,6] for references) shows that $w \leqq 0$ everywhere in $K(\alpha)$; the dominant term $|w(a)|$ in inequality (42) just gives the integrability condition $u \not\equiv -\infty$.

Remark 6. The constant M_3 in (42) is proportional to B. Thus, for $\rho > \rho_1$, theorem I assets that

$$\overline{\lim_{R\to\infty}}\ R^{-\rho-m+2} \int_{K_R(\beta)} \Delta u \leqq C(\beta)B.$$

This estimate can be improved slightly to a bound analogous to that in Theorem I (§2).

Because of its importance, we restate Theorem 1 in terms of analytic functions.

THEOREM 2. *Suppose* f *is an analytic function in the cone* $K(\alpha)$ *of* $\mathbb{C}^n$ *such that* $f(0) \neq 0$ *and* f *has order* $\rho > \rho_1(\alpha)$ *and finite type. If* $\sigma_\beta(R)$ *denotes the area of the variety* $V \cap K_R(\beta)$ $(\beta < \alpha)$, *we have*

$$\overline{\lim_{R\to\infty}} \frac{\sigma_\beta(R)}{R^{\rho+2n-2}} \leq C(\beta)B$$

(B *is the constant appearing in the definition of the type of* f*)*

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